Mean Value Theorem (MVT)

$f: [a,b] \to \mathbb{R}$ cont., differentl. in (a,b)

$\Rightarrow \exists\ x_0 \in (a,b)$ s.t. $\dfrac{f(b)-f(a)}{b-a} = f'(x_0)$

Corollary 1: $f'(x) = 0$ in $(a,b) \Rightarrow f(x) = \text{const.}$

Corollary 2 $f'(x) = g'(x)\ \forall x \in (a,b) \Rightarrow f(x) = g(x) + C$

for some const. C

Def. f strictly increasing on $[a,b]$ if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

f increasing " " if $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$

same for decreasing

Corollary 3 If $f'(x) > 0\ \forall x \in (a,b) \Rightarrow f$ strictly increasing

" " ≥ 0 " " $\Rightarrow f$ increasing

same for decreasing (~ $f'(x) < 0$ resp $f'(x) \leq 0$)

(proof: straightforward consequence of MVT)

Observe: There exist functions f differentiable everywhere but f' is NOT contin. everywhere

HW Ch28.4: $f(x) = x^2 \sin \frac{1}{x}$
differentiable everywhere, but f' NOT cont. at $x=0$

nevertheless even in this case, an MVT holds for f'

__Theorem__ (MVT for derivative)
$f: (a,b) \to \mathbb{R}$ differentiable, $a < x_1 < x_2 < b$
If c number between $f'(x_1)$ and $f'(x_2)$
$\Rightarrow \exists\, x_0 \in (x_1, x_2)$ s.t. $f'(x_0) = c$

proof assume $f'(x_1) < c < f'(x_2)$ (similar proof for $f'(x_2) < c < f'(x_1)$)

$\Rightarrow \quad f'(x_1) - c < 0 < f'(x_2) - c$

$\Updownarrow$

let $g(x) = f(x) - cx$

$\Rightarrow \quad \boxed{g'(x_1) < 0 < g'(x_2)}$

g continuous $\Rightarrow$ it has min and max in $[x_1, x_2]$

need to rule out that min could be at the boundary points (i.e. at x_1 or at x_2)

know: $g'(x_1) < 0$ $\qquad$ $g'(x_2) > 0$

$$\Rightarrow \lim_{\substack{y \to x_1 \\ y > x_1}} \frac{g(y) - g(x_1)}{\underbrace{y - x_1}_{>0}} = g'(x_1) < 0$$

$\Rightarrow g(y) - g(x_1) < 0$ for y close to x_1

$\Rightarrow g$ does not have minimum at x_1

same way: rule out that g has min. at x_2

$\Rightarrow g$ has min. at point $x_0 \in (x_1, x_2)$

$\Rightarrow g'(x_0) = 0 \Rightarrow f'(x_0) - c = 0$

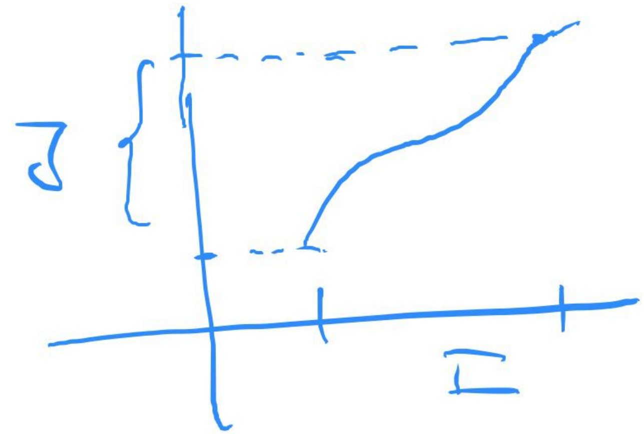
$\Rightarrow$ claim.

Derivative of inverse function:

$f: I \to \mathbb{R}$ $\quad$ I interval

f one-to-one and differentiable

have seen f either strictly increasing
or " decreasing



$J = f(I)$

can define function

$f^{-1}: J \to I$

apply f

$f^{-1}(f(x)) = x \quad \forall x \in I$

If $y = f(x)$ we have

$f(f^{-1}(y)) = y \quad \forall y \in J$

Questions: • Is f^{-1} differentiable ?

• Calculate $(f^{-1})'(y)$

Assume it is differentiable at, say, $y_0 \in J$ $\quad y_0 = f(x_0)$

apply chain rule.

$$f^{-1}(f(x)) = x$$

differentiate, using chain rule:

$$1 = (f^{-1})'(\underbrace{f(x_0)}_{y_0}) \cdot f'(x_0)$$

If $f'(x_0) \neq 0$

$$\Rightarrow \boxed{(f^{-1})'(y_0) = \frac{1}{f'(x_0)}}$$

This is indeed true (Theorem later).

Can be used to calculate derivatives of certain functions.

Example 1 $\quad f(x) = x^n \quad , n > 0$

$$f: (0, \infty) \longrightarrow (0, \infty)$$

$$f'(x) = n x^{n-1} > 0 \quad \text{for } x > 0$$

$\Rightarrow$ one-to-one.

$\Rightarrow \quad f^{-1}(x^n) = x \qquad$ well-defined on $(0, \infty)$

Let $\boxed{y = x^n} \Longleftrightarrow \boxed{x = y^{1/n}}$

$\Rightarrow \quad f^{-1}(y) = y^{1/n}$

$$\Rightarrow \quad (f^{-1})'(y) = \frac{1}{f'(x)} = \frac{1}{n x^{n-1}} = \frac{1}{n (y^{1/n})^{n-1}}$$

$$= \frac{1}{n} y^{\frac{1-n}{n}}$$

$$= \frac{1}{n} y^{\frac{1}{n} - 1}$$

$\Rightarrow \quad (f^{-1})(y) = y^{1/n}$ has derivative $\frac{1}{n} y^{\frac{1}{n} - 1}$

Example 2

$f(x) = \sin x$

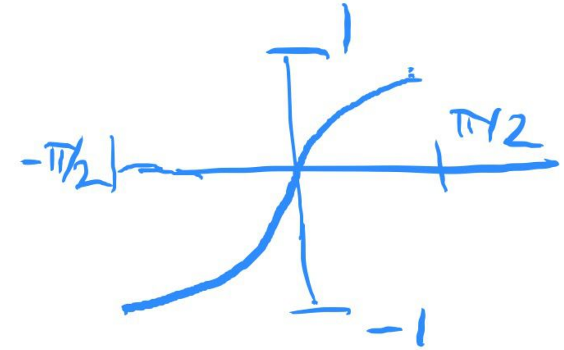
defined on interval $[-\pi/2, \pi/2]$

it has inverse function

$g(y) = \arcsin(y)$ defined on $[-1,1]$

$\shortparallel$

$f^{-1}(y)$



$$g'(y) = \frac{1}{f'(x)} = \frac{1}{\cos x}$$

need to express in terms of $y = \sin x$

$\cos x \geq 0$ for $-\pi/2 \leq x \leq \pi/2$

$$= \frac{1}{\sqrt{1-\sin^2 x}}$$

$$= \frac{1}{\sqrt{1-y^2}}$$

hence: The derivative of $\arcsin x$ is equal to $\dfrac{1}{\sqrt{1-x^2}}$

All our calculations hold because of the following

Theorem Assume

- $f : I \to \mathbb{R}$ 1-1 and differentiable
- $f'(x_0) \neq 0$

$\Rightarrow$ The inverse function $f^{-1} : J = f(I) \to \mathbb{R}$ is differentiable at $y_0 = f(x_0)$ and its derivative is given by

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$$

Remark: Main point to prove: f^{-1} is differentiable at y_0

have already seen: formula of $(f^{-1})'$ would follow from chain rule.